# Estimating the Central Direction of Random Rotations in $SO(3)$

Bryan Stanfill, Ulrike Genschel, Heike Hofmann, Iowa State University, Department of Statistics

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## Introduction

Various estimators of the central direction for rotations in $SO(3)$ have been proposed in the literature but little attention has been paid to comparing these estimators with respect to statistical properties such as bias and sampling variability. We compare four estimators across three commonly used distributional models from the family of uniform-axis random spin distributions. Three of these estimators have previously been studied whereas we adapt Fisher's spherical median to rotation matrices for the first time. Through the means of a simulation study, we show that no estimator performs optimal across all distributions but rather that the performance of the estimators depends on underlying distribution characteristics.

## $SO(3)$ Data and its Uses

Orientation data in three dimensions can be represented by $3 \times 3$ rotation matrices that are orthogonal with determinant 1. The set of these matrices is commonly called the rotation group $SO(3)$. For our study we consider three popular symmetric rotational distributions for various values of the circular spread, $\nu$. This type of data is used by computer scientists interested in computer vision, and kinesiologists in modeling the movement of joints.

## Estimators for $SO(3)$ Data

We consider four estimators of the central direction. They are means and medians based on the Euclidean and Riemannian distances in $\boldsymbol{SO(3)}$ defined below, respectively:

$$d_E(\boldsymbol{o}_1, \boldsymbol{o}_2) = \|\boldsymbol{o}_1 - \boldsymbol{o}_2\|_F, \tag{1}$$

$$d_R(\boldsymbol{o}_1, \boldsymbol{o}_2) = \frac{1}{\sqrt{2}} ||\mathrm{Log}(\boldsymbol{o}_1^\top \boldsymbol{o}_2)||_F. \tag{2}$$

For a sample $\boldsymbol{o}_1, \ldots, \boldsymbol{o}_n$ of random rotations the four estimators are defined as follows:

| Name | | Definition |
|---|---|---|
| Fisher's Median | $^*\widetilde{\boldsymbol{S}}_E$ | $\min_{\boldsymbol{S}} \sum_{i=1}^n d_E(\boldsymbol{o}_i, \boldsymbol{S})$ |
| Projected Mean | $\widehat{\boldsymbol{S}}_E$ | $\min_{\boldsymbol{S}} \sum_{i=1}^n d_E^2(\boldsymbol{o}_i, \boldsymbol{S})$ |
| Geodesic Median | $\widetilde{\boldsymbol{S}}_R$ | $\min_{\boldsymbol{S}} \sum_{i=1}^n d_R(\boldsymbol{o}_i, \boldsymbol{S})$ |
| Geometric Mean | $\widehat{\boldsymbol{S}}_R$ | $\min_{\boldsymbol{S}} \sum_{i=1}^n d_R^2(\boldsymbol{o}_i, \boldsymbol{S})$ |

Table 1: Estimators of the central direction.

* indicates the estimator we propose for the first time.

## $\widetilde{S}_E$ in Detail

Identification of the median in $\mathbb{R}^d$, often referred to as the Weber problem, was famously solved by the Weiszfeld algorithm [3]. We modify this algorithm by evaluating only proper rotations, i.e., $\boldsymbol{o} \in SO(3)$. Any rotation can serve as the initial starting rotation, $\widetilde{\boldsymbol{S}}^{(0)}$, so without loss of generality we choose $\widehat{\boldsymbol{S}}_E$ to speed convergence. Let $l$ denote the iteration of the algorithm, beginning with $l = 1$:

1. Set $\boldsymbol{s}_i = \boldsymbol{o}_i - \widetilde{\boldsymbol{S}}^{(l-1)}$.
2. Calculate
$$\bar{o}_W = \frac{\sum_{i=1}^n \boldsymbol{o}_i / ||\boldsymbol{s}_i||_F}{\sum_{i=1}^n 1/||\boldsymbol{s}_i||_F}.$$
3. Define $\widetilde{\boldsymbol{S}}^{(l)}$ to be the projection of $\bar{o}_W$ into $SO(3)$.
4. Repeat this process until $\epsilon > ||\widetilde{\boldsymbol{S}}^{(l-1)} - \widetilde{\boldsymbol{S}}^{(l)}||_F$

## Application

To evaluate the performance of the estimators on real data, we use the drilling data found in [2]. The goal of the study was to compare the variability between and within the orientations of 8 subjects (5 observations each) whilst drilling (Figure 1).

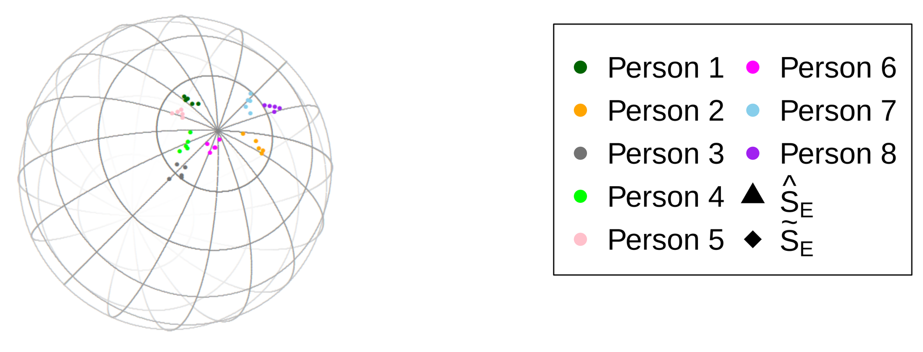


Figure 1: Data from [2].

Figures 2 and 3 display $\widehat{\boldsymbol{S}}_E$ and $\widetilde{\boldsymbol{S}}_E$ with 95% bootstrap intervals, respectively, for each individual and the combined data. Since the data are highly concentrated, the difference between $d_R$ and $d_E$ is small so only $d_E$ based estimators are considered here.

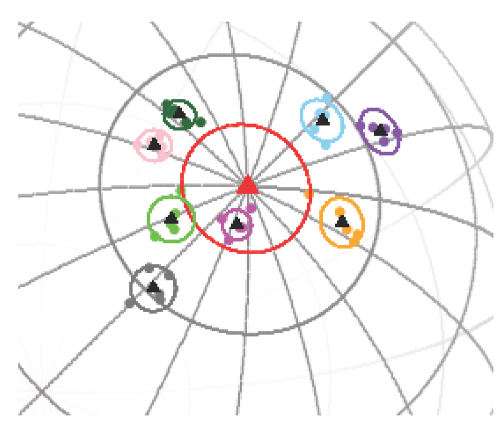

Figure 2: $\widehat{S}_E$ estimates with 95% CIs.

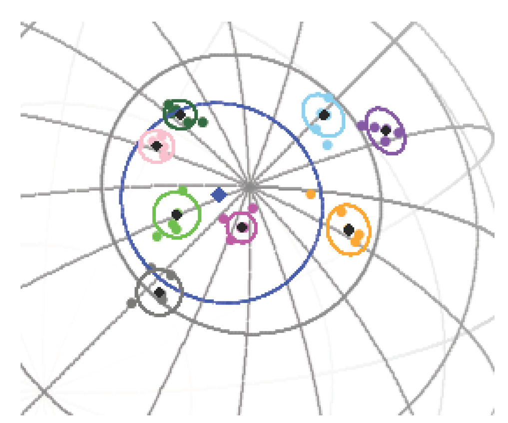

Figure 3: $\widetilde{S}_E$ estimates with 95% CIs.

$\widehat{S}_E$ performs better at detecting the large between variability relative to within variability.

## Simulation Results

In the following we summarize and present a subset of the simulation study findings. For each set of the simulated samples the estimation error is determined using the Riemannian distance. The error distributions for each estimator with $\nu = 0.25$ and $\nu = 0.75$ can be found in Figures 4 and 5 respectively.

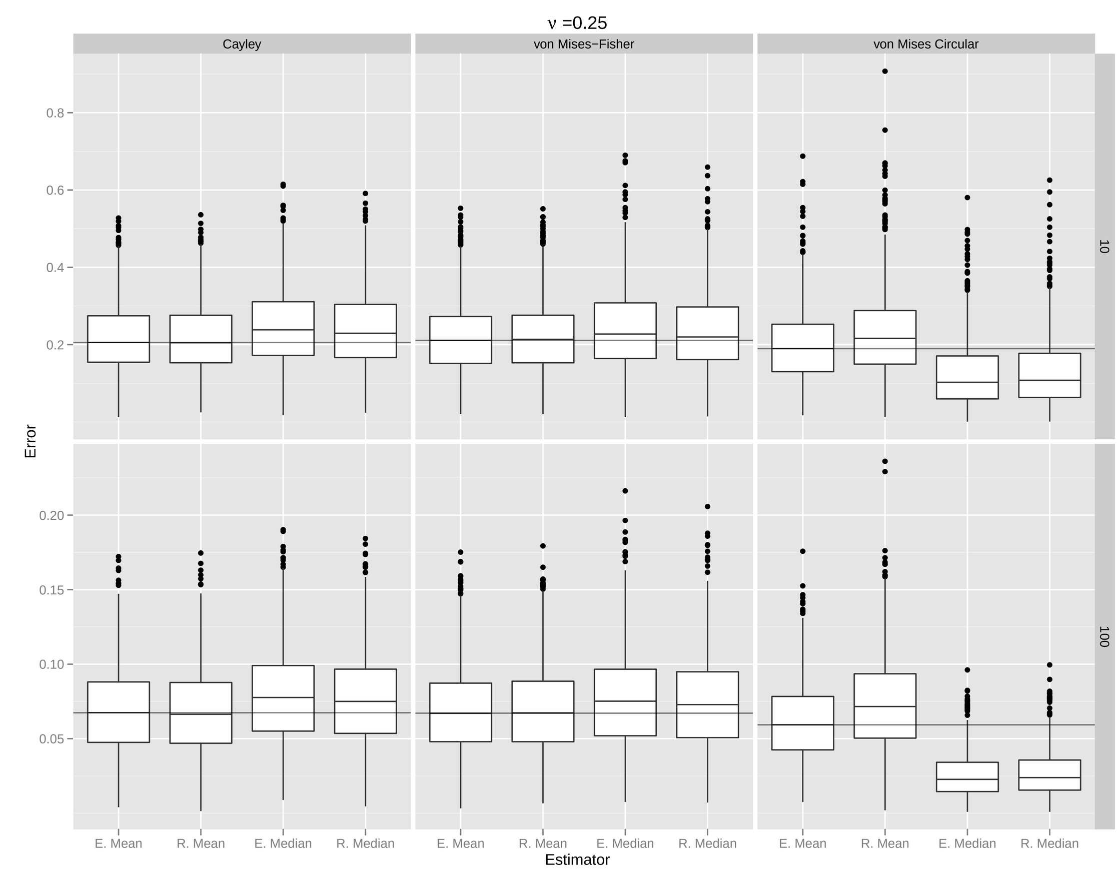


Figure 4: Error distributions of the Estimators for $\nu = 0.25$ and $n = 10, 100$

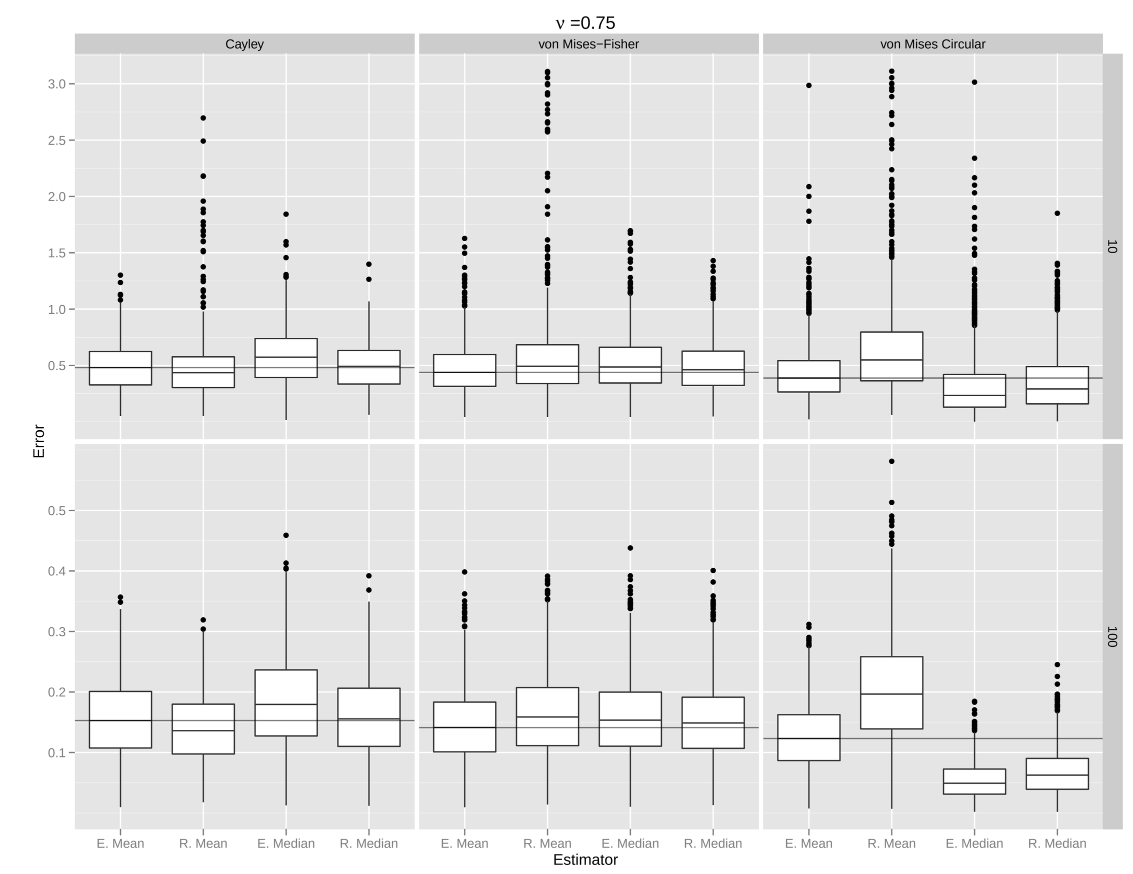


Figure 5: Error distributions of the Estimators for $\nu = 0.75$ and $n = 10, 100$

To further investigate the performance of $\widehat{\boldsymbol{S}}_E$ and $\widehat{\boldsymbol{S}}_R$ in Figure 5, consider Figures 6-9 which plot both estimators for the von Mises Fisher and Cayley distributions for two values of $\nu$.

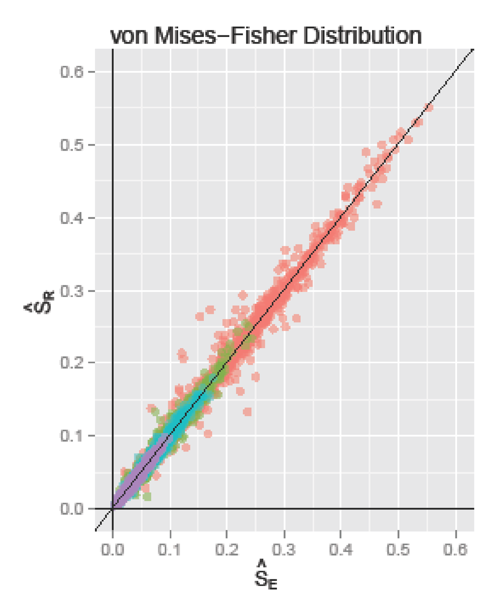


Figure 6: Fisher, $\nu$=0.25

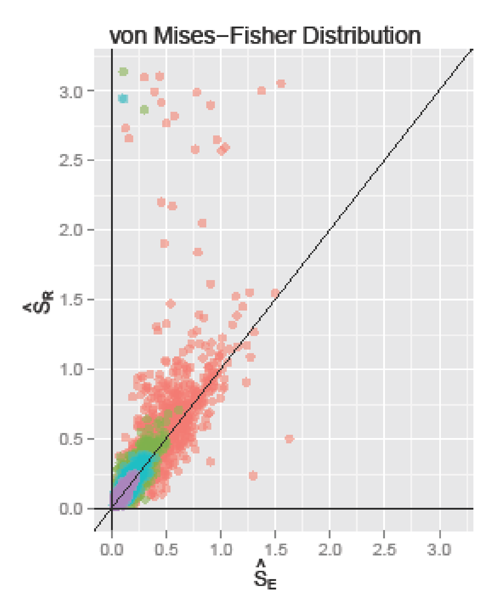


Figure 7: Fisher, $\nu$=0.75

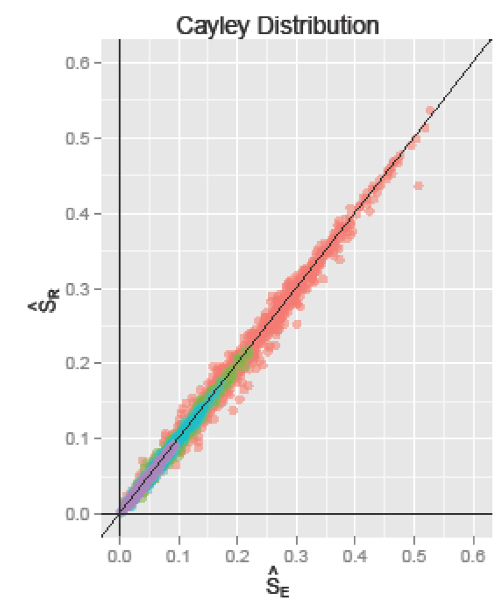


Figure 8: Cayley, $\nu$=0.25

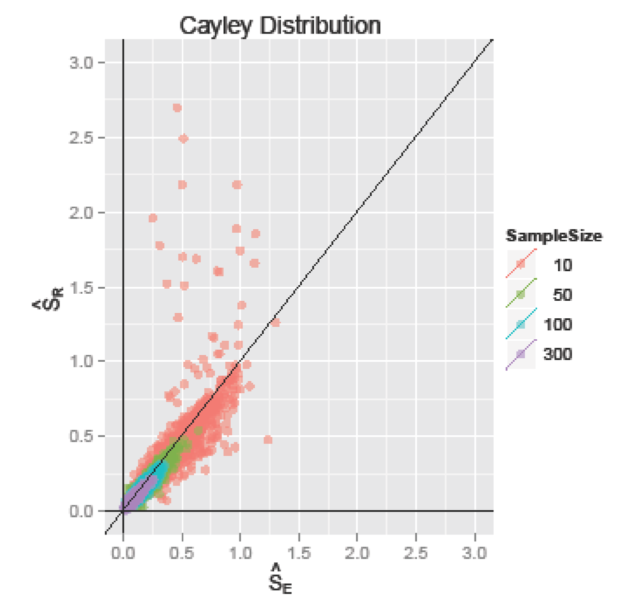


Figure 9: Cayley, $\nu$=0.75

For highly concentrated data the estimator tend to agree (Figures 6 and 8). For more dispersed data (Figures 7 and 9) the results differ by distribution, but $\widehat{\boldsymbol{S}}_E$ proves more resistant to extreme observations as indicated by the large number of points in the upper left corner of both figures. This is supported by the observation that on average $\widehat{\boldsymbol{S}}_E$ is preferred for the von Mises Fisher distribution (heavy tails), whereas the opposite is true for the Cayley distribution (light tails).

## Further Research

We plan to support these empirical findings with theoretical arguments and extend our work to estimation of the concentration parameter $\kappa$, which has also been given little attention in the literature. An R package is under development that will generate data from various rotational distributions and provide efficient algorithms for computing estimators of the central direction.

## References


[1] Moakher, M., *Means and averaging in the group of rotations*, SIAM Journal on Matrix Analysis and Applications **24(1)**: 1–16, 2003.

[2] Rancourt, D. and Rivest, L.P. and Asselin, J., *Using orientation statistics to investigate variations in human kinematics*, Journal of the Royal Statistical Society: Series C (Applied Statistics) **49(1)**: 81–94, 2000.

[3] Weiszfeld, E., *Sur le point pour lequel la somme des distances de n points dennï£¡es est minimum*, Tï£¡ohoko Mathematics Journal **43**: 355–386, 1937.