Confidence Regions for the Central Orientation of Random Rotations

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Introduction

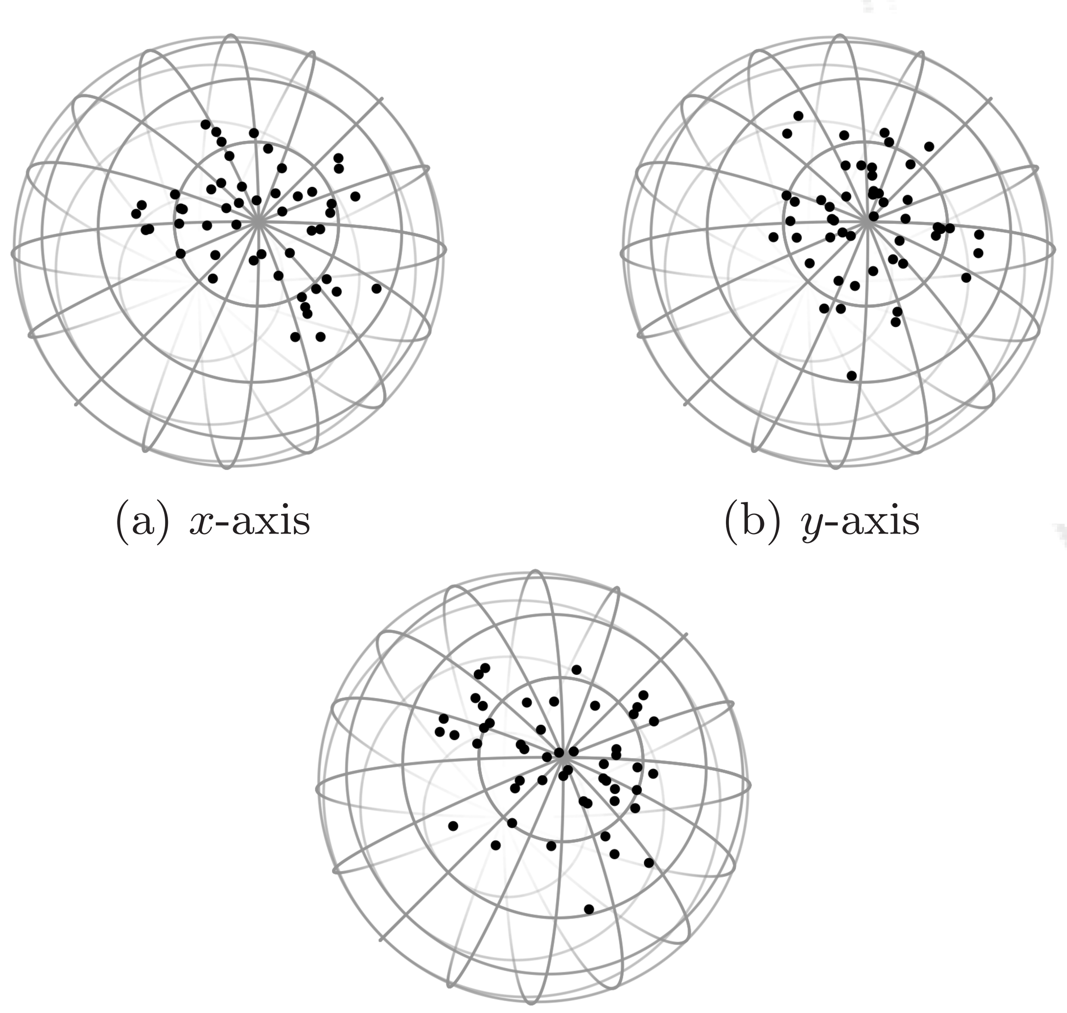
Though point estimation for random rotations has received considerable attention (see [3] for a review), inferential techniques are more limited. Here we derive the limiting distribution for extrinsic L_p estimators, which can be used to form asymptotic confidence regions (CR). Additionally we propose a nonparametric bootstrap CR that achieves close to nominal coverage rates for small n. In a simulation study we compare our methods to already existing ones in the literature.

$SO(3)$ Data and its Uses

- Orientation data in three dimensions can be represented by 3×3 orthogonal matrices with determinant one; the group of all such matrices is the rotation group denoted $SO(3)$.
- Some areas of application: structural geology, kinematics, material sciences, computer vision
- We consider the location model in $SO(3)$ given by

$$\boldsymbol{R}_i = \boldsymbol{S}\boldsymbol{E}_i \qquad (1)$$

where $\boldsymbol{E}_i \in SO(3)$ are i.i.d. directionally symmetric perturbations of the central orientation $\boldsymbol{S} \in SO(3)$.
- (1) is analogous to the model $y_i = \mu + e_i$ on $\mathbb{R}$ for $e_i \overset{iid}{\sim} (0, \sigma^2)$



(a) x-axis (b) y-axis (c) z-axis

Figure 1: A random sample from the $SO(3)$ location model with $\boldsymbol{S} = \boldsymbol{I}_{3\times 3}$ and $\boldsymbol{E}_i \overset{iid}{\sim}$Cayley($\kappa = 50$)

- Two approaches to estimating $\boldsymbol{S}$ in (1): the extrinsic approach based on the Euclidean distance and the intrinsic approach which makes use of innate $SO(3)$ topology by using Riemannian distance, see [3] for more details.

Extrinsic L_p Estimators

- Extrinsic (projected) estimators are based on the Euclidean distance metric defined for rotations $\boldsymbol{R}_1, \boldsymbol{R}_2 \in SO(3)$ as

$$d_E(\boldsymbol{R}_1, \boldsymbol{R}_2) = \|\boldsymbol{R}_1 - \boldsymbol{R}_2\|_F$$

where $\|\boldsymbol{A}\|_F^2 = \mathrm{tr}(\boldsymbol{A}^\top \boldsymbol{A})$ is the Frobenius norm of the matrix $\boldsymbol{A}$ and $\mathrm{tr}(\cdot)$ is the matrix trace

- The projected median is defined as

$$\widehat{\boldsymbol{S}}_1 = \underset{\boldsymbol{S}\in SO(3)}{\arg\min} \sum_{i=1}^n d_E(\boldsymbol{R}_i, \boldsymbol{S})$$

- The projected mean is defined as

$$\widehat{\boldsymbol{S}}_2 = \underset{\boldsymbol{S}\in SO(3)}{\arg\min} \sum_{i=1}^n d_E^2(\boldsymbol{R}_i, \boldsymbol{S}) = \underset{\boldsymbol{S}\in SO(3)}{\arg\max}\, \mathrm{tr}(\boldsymbol{S}^\top \overline{\boldsymbol{R}})$$

where $\overline{\boldsymbol{R}} = \sum_{i=1}^n \boldsymbol{R}_i / n$.

Large Sample Theory

Proposition: Assume $\boldsymbol{R}_1, \boldsymbol{R}_2, \ldots, \boldsymbol{R}_n$ are a sample of i.i.d. observations from a rotationally symmetric location model with central orientation $\boldsymbol{S}$, $\widehat{\boldsymbol{S}}_P = \arg\min_{\boldsymbol{S}\in SO(3)} \sum_{i=1}^n d_E^p(\boldsymbol{R}_i, \boldsymbol{S})$ is a consistent estimator for $\boldsymbol{S}$ and $\hat{\boldsymbol{h}}_p$ satisfies $\exp[\boldsymbol{\Phi}(\hat{\boldsymbol{h}}_p)] = \boldsymbol{S}^\top \widehat{\boldsymbol{S}}_P$. Then

$$\sqrt{n}\hat{\boldsymbol{h}}_p \xrightarrow{d} MVN_3\left(\boldsymbol{0}, \frac{c}{2d^2}I\right)$$

as $n \to \infty$ or equivalently

$$\frac{2nd^2}{c}\|\hat{\boldsymbol{h}}_p\| \to \chi_3^2 \qquad (2)$$

as $n \to \infty$. When $p = 1$, $\widehat{\boldsymbol{S}}_P = \widehat{\boldsymbol{S}}_1$,

$$c = \frac{1}{6}E[1+\cos(r)] \quad \text{and} \quad d = \frac{1}{12}E\left[\frac{1+3\cos(r)}{\sqrt{1-\cos(r)}}\right];$$

assuming that $\widehat{\boldsymbol{S}}_1 \neq \boldsymbol{R}_i$ for all i guarantees d is defined. When $p = 2$, $\widehat{\boldsymbol{S}}_P = \widehat{\boldsymbol{S}}_2$,

$$c = \frac{2}{3}E[1-\cos^2(r)] \quad \text{and} \quad d = \frac{1}{3}E[1+2\cos(r)].$$

From (2), the set of rotations $\boldsymbol{R}$ that satisfy

$$\frac{2n\hat{d}^2}{\hat{c}} d_R(\widehat{\boldsymbol{S}}_P, \boldsymbol{R})^2 < \chi^2_{3,1-\alpha}$$

form a $100(1-\alpha)$% CR for the central orientation $\boldsymbol{S}$ where $\hat{c}$ and $\hat{d}$ are consistent estimators of c and d, respectively.

Simulation Study

- The convergence rate of (2) is different for $\widehat{\boldsymbol{S}}_1$ and $\widehat{\boldsymbol{S}}_2$, see Figure 2 (right).
- Different c, d values for $\widehat{\boldsymbol{S}}_1$ and $\widehat{\boldsymbol{S}}_2$ result in different asymptotic variances, Figure 3 (below) illustrates the effect on CR coverage rates.

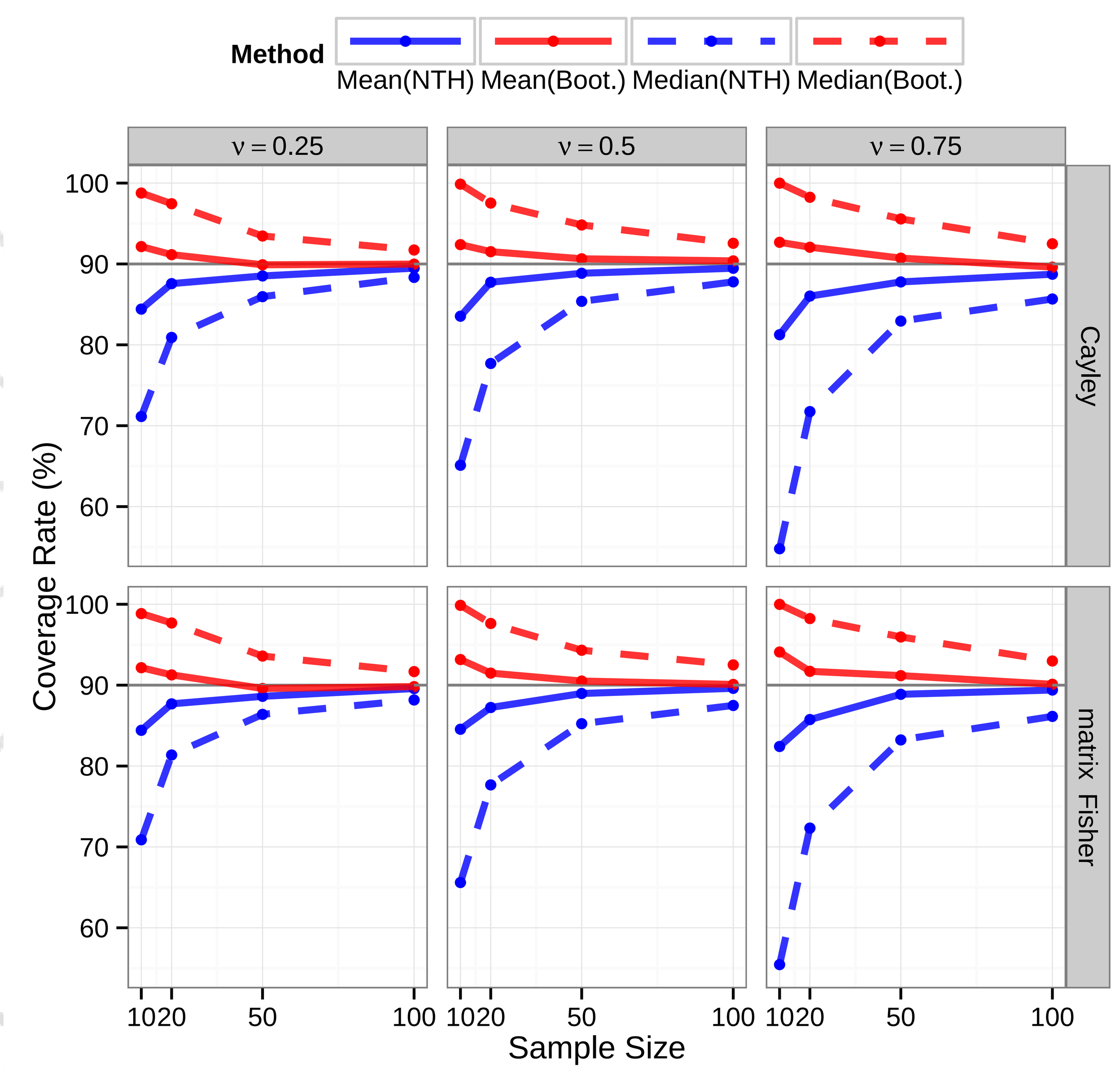


Figure 3: Coverage rates for 90% CRs using $\widehat{\boldsymbol{S}}_1$ (dashed) and $\widehat{\boldsymbol{S}}_2$ (solid)

- Existing CR methods based on $\widehat{\boldsymbol{S}}_2$ rely on eigenvector asymptotics, our result is moment-based. No other CRs centered at $\widehat{\boldsymbol{S}}_1$ exist.
- Figure 4 (right) compares existing methods (denoted "Eigen") to our method (denoted "Moment"); for normal theory see [1], for bootstrap see [2].

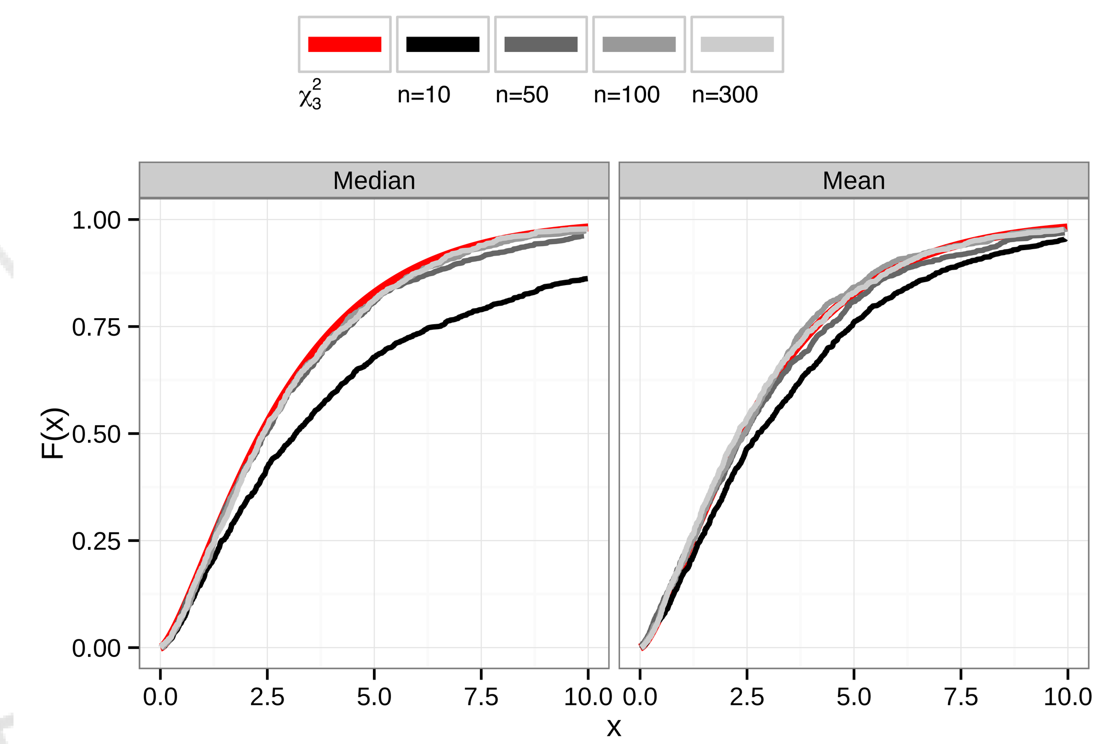


Figure 2: Critical value ECDF and theoretical limiting χ_3^2 distribution

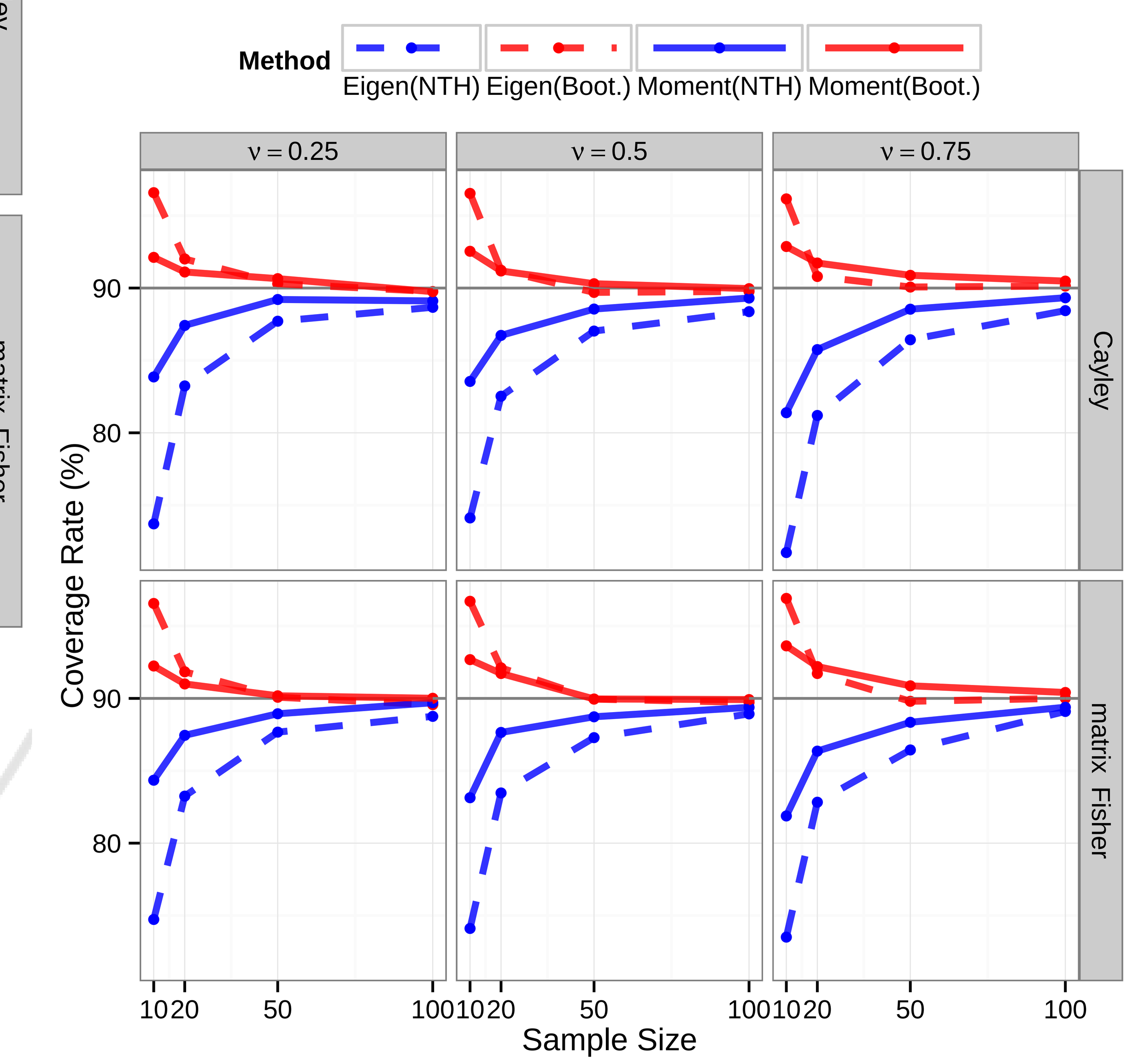


Figure 4: Coverage rates for 90% CRs using $\widehat{\boldsymbol{S}}_2$ based on new (solid) and existing methods (dashed)

Discussion

- $\widehat{\boldsymbol{S}}_1$ has a larger asymptotic variance than $\widehat{\boldsymbol{S}}_2$ for rotationally symmetric samples.
- Critical value based on $\widehat{\boldsymbol{S}}_1$ converges more slowly than for $\widehat{\boldsymbol{S}}_2$.
- All CR methods monotonically approach nominal coverage rate as n increases.
- Moment-based methods generally perform best, especially for small n.

Further Research

- Explore estimator and CR behavior under more general assumptions
- Extend the result to intrinsic estimators
- Quantify differences in extrinsic versus intrinsic estimation choice

References

[1] M. Prentice. A distribution-free method of interval estimation for unsigned directional data. *Biometrika*, 71(1):147–154, 1984.

[2] N. Fisher, P. Hall, B. Jing, and A. Wood. Improved pivotal methods for constructing confidence regions with directional data. *JASA*, 91(435):1062–1070, 1996.

[3] B. Stanfill, U. Genschel and H. Hofmann. Point Estimation of the Central Orientation of Random Rotations. *Technometrics*, 55(4), 2013.